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NOTE ON THE COMPUTATION OF THE ABSOLUTE PERTURBATIONS OF A COMET.

By Prof. Ormond Stone, University of Virginia.

The large eccentricities of the orbits of the periodic comets have thus far rendered it impracticable to compute tables of their motions as has been done in the case of planets and their satellites. Hansen, however, by the introduction of subsidiary quantities known as *partial anomalies*, has made a decided advance toward a solution of the problem. Since the publication of Hansen's Memoir, Gyldén and others have made some important modifications of his methods. The only applications of the method as yet made, so far as I am aware, have been to the partial computation of the perturbations of Encke's comet, and the results reached, though interesting and valuable, can by no means be considered satisfactory.

In Hansen's method the orbit is divided into two parts, the points of division being at equal distances on either side of the perihelion. The partial anomaly which includes the perihelion is made a function of the eccentric anomaly, while that containing the aphelion is made a function of the true anomaly. Afterward, by the application of special devices, the orbit may be divided into a greater number of parts.

The following is suggested as possibly offering some advantages:—

Put $\varepsilon = \eta_0 + \eta$, where η_0 is put successively equal to different values of the eccentric anomaly (ε) corresponding with points in the orbit selected arbitrarily In general, these will be selected so as to divide the orbit into equal portions with regard to ε. Put also $\sin \operatorname{am} z = \lambda(z)$, $\cos \operatorname{am} z = \mu(z)$, $\Delta \operatorname{am} z = \nu(z)$; then, within equal limits on each side of η_0, we may write

$$\sin\eta = k\,\lambda(z),$$

$$\cos\eta = \nu(z),$$

$$\frac{d\eta}{dz} = k\,\mu(z),$$

$$k = \text{modulus}.$$

If, now, we put

$$l_{\prime} = k \sin\varphi \sin\eta_0, \quad m_{\prime} = -\sin\varphi\cos\eta_0,$$
$$l_{\prime\prime} = -k\sin\eta_0, \quad m_{\prime\prime} = \cos\eta_0,$$
$$l_{\prime\prime\prime} = k\cos\varphi\cos\eta_0, \quad m_{\prime\prime\prime} = \cos\varphi\sin\eta_0;$$

we may readily obtain the following general equations:—

$$\frac{r}{a} = 1 + l_{\prime}\lambda(z) + m_{\prime}\nu(z),$$
$$\frac{r}{a}\cos f = l_{\prime\prime}\lambda(z) + m_{\prime\prime}\nu(z) - e,$$
$$\frac{r}{a}\sin f = l_{\prime\prime\prime}\lambda(z) + m_{\prime\prime\prime}\nu(z),$$
$$\frac{dg}{dz} = k\frac{r}{a}\mu(z);$$

where, as may be seen at a glance, $e = \sin\varphi$ is the eccentricity, a the semi major axis, r the radius vector, f the true anomaly, and g the mean anomaly.

If the intervals between the different values of η_0 be the same, k will be the same in all parts of the orbit, and a single development only will be needed of the quantities $\lambda(z)$, $\mu(z)$, and $\nu(z)$.

ON THE DESIGN OF STEPPED PULLEYS FOR LATHE GEARS.

By Prof. Wm. M. Thornton, University of Virginia.

The problem to be solved in this case is the determination of the diameters of a set of pairs of pulleys which will transmit the motion of the shaft to the lathe spindle with given angular velocity ratios by a belt of constant length. The fundamental formulæ for the solution of this problem are easily written down. We have

$$L = (\pi + 2\theta)R + (\pi - 2\theta)r + 2c\cos\theta,$$
$$\sin\theta = \frac{R - r}{c},$$
$$n = \frac{R}{r};$$

where L is the constant length of belt,

c the distance between the pulley centres,

n the given angular velocity ratio,

R the radius of the larger pulley,

r the radius of the smaller pulley,

θ the angle between the line of centres and the straight belt.

The data are L, c, n; R, r are required.

The solution of this group of equations is effected in the present paper by using the auxiliary quantity $x = \sin\theta$ to deduce a relation between the known ratio $\frac{L}{c}$, the given velocity ratio n, and x. If x be found from this relation we can obtain R and r at once from the formulæ

$$r = \frac{x}{n - 1}\,c, \quad R = \frac{nx}{n - 1}\,c.$$

To carry out this solution we reduce the expression for $\frac{L}{c}$ to the form

$$\frac{L}{c} = 2\,(\theta\sin\theta + \cos\theta) + \frac{n + 1}{n - 1}\,\pi\sin\theta,$$

and in it put

$$\begin{aligned}
\sin\theta &= x,\\
\theta &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots,\\
\cos\theta &= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \dots,\\
L &= 2c + 2\pi R_0,\\
\rho &= \frac{n - 1}{n + 1};
\end{aligned}$$

R_0 representing the radius of either of the pair of equal pulleys. After suitable reductions are made we find

$$x = 2\rho\frac{R_0}{c} - \frac{\rho}{\pi}x^2\left\{1 + \frac{1}{12}x^2 + \frac{1}{40}x^4 + \frac{5}{448}x^6 + \dots\right\}.$$

If we neglect the variable terms in the bracket and put for x on the right the first approximation $2\rho\frac{R_0}{c}$, we get the quite close approximation

$$\xi = 2\rho\frac{R_0}{c} - \frac{\rho}{\pi}\left(2\rho\frac{R_0}{c}\right)^2.$$

Putting ξ for x in the parentheses we get a still closer approximation, which may in turn be used as the basis for a closer one; and so on, until the primitive and the derived values of x agree to the requsite number of decimal places.

For example, take the case in which $c = 120'$ and $R_0 = 12'$. We find

$$L = 240 + 24\pi = 315{,}40,$$

and for the radii of the pulleys which must be used with the same belt to communicate the angular velocity ratio 3 : 2 we get

$$\begin{aligned} \xi &= 2 \cdot \frac{1}{5} \cdot \frac{1}{10} - 0{,}3183 \cdot \frac{1}{5} \cdot \left(\frac{1}{25}\right)^2, \\ &= 0{,}0399, \\ \therefore\ x &= 0{,}0399, \\ r &= 9{,}58, \\ R &= 14{,}37. \end{aligned}$$

The error committed in using ξ for x is obviously less than

$$\frac{4}{3\pi}\rho^5\left(\frac{R_0}{c}\right)^4,$$

and may therefore be disregarded if the value of this expression is less than a half unit of the last place retained in the value of x. Thus in the example above this error is less than $136 : 10^{10}$, so that if it were desired we could carry the value of ξ to seven decimal places and it would agree to the last place inclusive with the value of x. We should get thus

$$x = 0{,}0398981.$$

Such closeness of approximation of course far exceeds the capacities of the machine shop. It will be found that in almost every case of actual practice the approximation

$$x = 2\rho\frac{R_0}{c} - \frac{\rho}{\pi}\left\{2\rho\frac{R_0}{c}\right\}^2$$

will furnish values of r and R exact to the nearest hundredth of an inch.

A convenient method of determining with rapidity and with an exactness sufficient for practical purposes the value of x corresponding to given values of ρ and $\frac{R_0}{c}$ is to construct to a large scale the curve

$$y = x^2 + \frac{1}{12}x^4 + \frac{1}{40}x^6 + \frac{5}{448}x^8 + \ldots,$$

and mark its point of intersection with the straight line

$$\frac{x}{\rho}+\frac{y}{\pi}=2\frac{R_0}{c}.$$

The abscissa of this point is the required value of x and R, r are then found as above. The curve can be constructed once for all and measures can afterward be taken directly from the sheet, the straight line being drawn in by its axial intercepts.

The methods heretofore in use for the design of these pulleys proceed by assuming values for the difference $R-r$ instead of for the ratio $R:r$. Sang, for example, calculates the diameter of a circle equal in length to the belt ($B=\frac{L}{\pi c}$), the distance between the centres of the pulleys being the linear unit, and using as argument the difference $R-r=x$ computes the differences

$$B-2r=\frac{2}{\pi}\left\{1+\frac{1}{2}x^2+\frac{1}{24}x^4+\ldots\right\}+x,$$
$$B-2R=\frac{2}{\pi}\left\{1+\frac{1}{2}x^2+\frac{1}{24}x^4+\ldots\right\}-x.$$

These differences being tabulated for equidistant values of x, we get from the table by interpolation $B-2r, B-2R$; whence r, R.* Culmann's method is based on the equivalent formulæ

$$2r=B-\frac{2}{\pi}\left[x\sin^{-1}x+\sqrt{(1-x^2)}\right]-x,$$
$$2R=B-\frac{2}{\pi}\left[x\sin^{-1}x+\sqrt{(1-x^2)}\right]+x;$$

which are used as the foundation of a grapical process for finding from a given x the values of $2R-x$ and $2r+x$.† The objection to all such methods is the fact that the determination ought to be based in all cases on the ratio of the radii.

*Spon's *Encyclopedia*: Belts.

†Cf. Moll u. Reuleaux, *Constructionslehre*, p. 321.

A DISCUSSION OF THE EQUATION OF THE SECOND DEGREE IN TWO VARIABLES.*

By Prof. O. H. Mitchell, Marietta, Ohio.

1. The equation of a conic section is by Boscovich's definition

$$(x - x')^2 + (y - y')^2 = e^2(x\lambda + y\mu - p)^2, \tag{A}$$

where (x', y') is the focus, e is a constant, and $x\lambda + y\mu - p = 0$ is the normal equation of the directrix. Multiplying through by ρ and assuming

$$\rho(1 - e^2\lambda^2) = a, \tag{1}$$
$$-\rho e^2\lambda\mu = h, \tag{2}$$
$$\rho(1 - e^2\mu^2) = b, \tag{3}$$
$$\rho(e^2p\lambda - x') = g, \tag{4}$$
$$\rho(e^2p\mu - y') = f, \tag{5}$$
$$\rho(x'^2 + y'^2 - e^2p^2) = c, \tag{6}$$

equation (A) becomes

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{B}$$

The co-efficients a, b, c, f, g, h are not conditioned by these equations (1) to (6), as there are six unknowns in the equations. Solving for these unknowns, we get

$$\left.\begin{aligned}
e^2 &= \frac{2R}{R+s}, \\
\mu &= +\sqrt{\frac{R+d}{2R}}, \\
\lambda &= \theta\sqrt{\frac{R-d}{2R}}, \\
\lambda\mu &= -\frac{h}{R}, \\
p &= \frac{f^2 + g^2 - \frac{1}{2}c(R+s)}{R(f\mu + g\lambda) \mp \sqrt{-R\Delta}} \\
&= \frac{2f\mu + 2g\lambda}{R-s} \pm \sqrt{\frac{-4\Delta}{R(R-s)^2}} \\
&= p' \pm p'', \text{ say,}
\end{aligned}\right\} \tag{7}$$

*An abstract of the following paper was presented to the University Mathematical Society of Baltimore in March, 1883, and it has lately been suggested to me that it might be worth while to publish it more in detail for the benefit of a more general class of readers. The method of discussion, though quite far removed from what may be designated as the modern methods, is perhaps the most elementary that can be given and for some purposes is by far the most convenient.

$$x' = e^2\left\{p\lambda - \frac{g}{R}\right\}$$
$$= \frac{bg - fh}{h^2 - ab} \pm \frac{\sqrt{-\tfrac{1}{2}\Delta(R-d)}}{h^2 - ab}, \qquad (7\ \textit{cont.})$$
$$y' = e^2\left\{p\mu - \frac{f}{R}\right\}$$
$$= \frac{af - gh}{h^2 - ab} \pm \frac{\sqrt{-\tfrac{1}{2}\Delta(R+d)}}{h^2 - ab};$$

where for brevity we write

$$s = a + b,$$
$$d = a - b,$$
$$R = \pm\sqrt{d^2 + 4h^2} = \pm\sqrt{s^2 + 4(h^2 - ab)},$$
$$\theta = \text{sign of } -\frac{h}{R},$$
$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2.$$

We thus see that for each of the two (real) values of R there is one real value of e^2; one real value of λ and of μ (and consequently one real value of the angle α); two values of p, one pair of values being real, the other pair being imaginary, unless (1) $\Delta = 0$, in which case the members of each pair are real and equal, or (2) $R = 0$, when all four values of p become infinite; and one value of x', y' corresponding to each value of p, real or imaginary according as p is real or imaginary. The two values of the angle α differ by $90°$. For let λ_1, μ_1 be the values of λ, μ obtained by substituting the positive value of R and λ_2, μ_2 be the other values, and let θ' be the sign of $-h$. Then

$$\lambda_1 = \theta'\mu_2 \text{ and } \lambda_2 = -\theta'\mu_1,$$

whence α_1 and α_2 differ by $90°$.

We have thus seen that the general equation of the second degree (B) is the equation of a conic section, and have found its foci, directrices, and constant ratio e in terms of its co-efficients. The results show that a conic has four directrices, two entirely real and parallel to each other, the other two perpendicular to the first pair at an imaginary distance from each other, unless (1) $\Delta = 0$, in which case the members of each pair are real and coincident, or (2) $R = 0$, when the directrices are all real and at an infinite distance from the origin. To each directrix corresponds one focus, real or imaginary according as the directrix is real or imaginary. To each pair of parallel directrices corresponds one value of e^2.

2. In what follows Δ is supposed to have been made negative when its value is not zero. The positive value of R, say R', will then give the real values of p,

x', y'. The equations of the four directrices having been found to be of the form

$$x\lambda + y\mu = p' \pm p'',$$

it follows that the equations of the two lines midway between the parallel directrices (the *axes* of the conic) are

$$x\lambda + y\mu = p', \tag{8}$$

where λ and μ have two values each. Substituting from (7) and reducing, (8) becomes

$$-2h\left(x + \frac{2g}{s - R}\right) + (R + d)\left(y + \frac{2f}{s - R}\right) = 0, \tag{9}$$

$$(R - d)\left(x + \frac{2g}{s - R}\right) - 2h\left(y + \frac{2f}{s - R}\right) = 0, \tag{10}$$

two forms of the same equation, of which one becomes indeterminate when $h = 0$. The positive value of R substituted either in (9) or in (10) gives the midway parallel to the real directrices (the *conjugate* axis), and the negative value of R gives likewise the midway parallel to the imaginary directrices, itself a real line (the *transverse* axis of the conic).

3. Now we might prove that the pair of real foci lie on the transverse axis by substituting their co-ordinates (7) in the equation of the axis, but the simplest way is as follows: If the transverse axis were taken as the axis of x and the conjugate as the axis of y, then we should have $\alpha_1 = 0$, $\alpha_2 = 90°$, that is $\mu_1 = 0$, $\mu_2 = 1$; but this implies $h = 0$. Furthermore the two values of p belonging to each pair of parallel directrices would then differ only in sign. This implies $f = 0$, $g = 0$ in the equation of the conic as referred to the new axes. The co-ordinates of the two real foci would then become (7)

$$x_1' = \pm e_1^2 p_1'' \lambda_1 = \pm e^2 p'',$$
$$y_1' = \pm e_1^2 p_1'' \mu_1 = 0;$$

whence it is seen that the two real foci lie on the transverse axis at equal distances from the *centre* (intersection of the axes) of the conic. In the same way the two imaginary foci may be said to lie on the conjugate axis at equal imaginary distances from the centre, for their co-ordinates are

$$x_2' = 0,$$
$$y_2' = \pm e_2^2 p_2''.$$

From this it follows, as also from the fact that the new equation is of the form

$$a'x^2 + b'y^2 + c' = 0, \tag{11}$$

that a conic is symmetrical with respect to both of its axes.

From (7) we see that the co-ordinates of the centre are

$$x_0 = \frac{bg - fh}{h^2 - ab}, \qquad (12)$$

$$y_0 = \frac{af - gh}{h^2 - ab}, \qquad (13)$$

a result we might also obtain by finding where the axes intersect.

4. From (11) we see that a conic crosses each axis in two points at equal distances from the centre. Call this distance A. From the definition of a conic we know that these two points of crossing (*vertices*) divide the distance F from each focus (on this axis) to its directrix internally and externally in the ratio e. Then

$$\frac{A - F}{p'' - A} = \frac{A + F}{p'' + A} = e,$$

whence we get $A = ep''$ and $F = Ae$.

If a perpendicular be erected to an axis at a focus, then the distance on it to the curve (the *semi-parameter*) is by definition e times the distance from the focus to the corresponding directrix. Hence

$$\text{semi-parameter} = e\,(p'' - e^2 p'') = A\,(1 - e^2).$$

Collecting these metrical results we have

$$\begin{aligned}
\text{Directrix to centre} &= \sqrt{\frac{-4\,\Delta}{R\,(R - s)^2}}, \\
\text{Semi-axis} = A &= \sqrt{\frac{-2\,\Delta}{(R - s)\,(h^2 - ab)}}, \\
\text{Focus to centre} &= \sqrt{\frac{-R\,\Delta}{(h^2 - ab)^2}}, \qquad (14) \\
\text{Focus to directrix} &= \sqrt{\frac{-4\,\Delta}{R\,(R + s)^2}}, \\
\text{Semi-parameter} &= \sqrt{\frac{-8\,\Delta}{(R + s)^3}}.
\end{aligned}$$

Each of these quantities has two values, a transverse and a conjugate value, arising from the positive and negative values of R respectively. Calling the two values of the semi-axis A_1 and A_2 respectively, we find $A_1^2 > A_2^2$ unless $R' = 0$, for these values may be written

$$A_1^2 = \frac{-8\,\Delta(s + R')}{(s^2 - R'^2)^2}, \quad A_2^2 = \frac{-8\,\Delta\,(s - R')}{(s^2 - R'^2)^2},$$

and $-\Delta$ is a positive quantity.

5. From (11) we get

$$\frac{x^2}{-\frac{c'}{a'}} + \frac{y^2}{-\frac{c'}{b'}} = 1,$$

where $-\frac{c'}{a'}$ and $-\frac{c'}{b'}$ are plainly the squares of the semi-axes of the conic (B). Hence the equation (B) when referred to the axes of the conic as axes of reference becomes

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1,$$

or

$$A_2^2x^2 + A_1^2y^2 - A_1^2A_2^2 = 0. \qquad (15)$$

If we apply to (15) formulæ (7) we may get the same metrical results in terms of A_1 and A_2.

6. Let us, now, see what the equation of the conic (B) becomes when we take the transverse axis as the axis of x and a line at right angles through the left hand vertex as the axis of y; and let us for convenience form its equation with respect to that focus and directrix which lie on opposite sides of the axis of y. We have then $\lambda = 1$, $\mu = 0$, $y' = 0$, $x' = e_1p_1''(e_1 - 1)$ and $p = p_1''(1 - e_1)$. We do not and need not know in regard to the signs of x' and p whether they are positive or negative, but they are opposite by hypothesis. By substituting the above values in (A) we get

$$(1 - e_1^2)\, x^2 + y^2 = 2e_1p_1''\,(1 - e_1^2)\, x,$$

or

$$\frac{s - R'}{s + R'}\, x^2 + y^2 = 2\eta\, P_1x, \qquad (16)$$

where η is the sign of $\frac{s - R'}{s + R'}$ and P_1 is the transverse semi-parameter.

Now it is clear that if the origin had been taken at the right hand transverse vertex, the resulting equation would have been the same as (16), except that the right hand member would have been negative; and in order to obtain the corresponding conjugate results, it is necessary only to substitute $-R'$ for $+R'$ in (16).

CLASSIFICATION OF CONICS.

In equations (15) and (16) as well as in the metrical results (14) we find involved only the quantities s, R', and Δ. The curves represented by (B) may therefore be classified according to the values of these three quantities.

When $R' < s$ numerically, then both values of e^2 are less than unity, one being positive, the other negative, and the curve is called an *ellipse*.

When $R' = s$, then the two values of e^2 are 1 and ∞, and the curve is called a *parabola*.

When $R' > s$ numerically, then both values of e^2 are greater than unity, and the curve is called an *hyperbola*.

Since $R' = \pm\sqrt{s^2 + 4(h^2 - ab)}$, the above three conditions are equivalent to the following: —

$h^2 - ab < 0$ for the *ellipse*,
$h^2 - ab = 0$ for the *parabola*,
$h^2 - ab > 0$ for the *hyperbola*;

except in the special case $a = b = h = 0$, when, although $h^2 - ab = 0$, $e^2 = \frac{0}{0}$. Equation (B) in this case degenerates to an equation of the first degree. Hence a straight line considered as a degenerate conic is to be regarded as belonging to either one of the three classes of conics.

Each of these three species of conics is subdivided according to the value of the product $R'sΔ$ as follows: —

CLASSIFICATION OF CURVES OF THE SECOND DEGREE.

		$h^2 - ab < 0$	$h^2 - ab = 0$	$h^2 - ab > 0$
$R's\Delta < 0$		Real, finite ellipse, axes unequal. $A_1 > A_2$.	True parabola.	Hyperbola with finite axes. $A_1^2 > -A_2^2$
$R's\Delta = 0$	$R's = 0$, $\Delta \neq 0$	$s\Delta < 0$, Real circle. $s\Delta > 0$, Imaginary circle. $A_1 = A_2$.	Conditions impossible.	Equilateral hyperbola. $A_1^2 = -A_2^2$
	$R's \neq 0$, $\Delta = 0$	Infinitesimal ellipse, axes unequal.	Two parallel lines. Real if $f^2 + g^2 - cs > 0$, Coincident, " $= 0$, Imaginary, " < 0.	Hyperbola with unequal infinitesimal axes. [Two intersecting lines.]
	$R's = 0$, $\Delta = 0$	Infinitesimal circle.	Single right line.	Equilateral hyperbola with infinitesimal axes. [Two lines at right angles]
$R's\Delta > 0$		Imaginary ellipse. $A_1 < A_2$.	Conditions impossible.	*Hyperbola with finite axes. $A_1^2 < -A_2^2$.

[TO BE CONTINUED.]

NOTES ON GAUSS'S THEORIA MOTUS, SECTION 114.

By Mr. A. Hall, Jr., Washington, D. C.

In section 114 of the *Theoria Motus* we have the following expression:—

$$\frac{(\mathrm{o}.2.\mathrm{II})\,\delta + (\mathrm{O}.2.\mathrm{II})\,D}{(\mathrm{o}.1.\mathrm{I})\,\delta + (\mathrm{O}.1.\mathrm{I})\,D} \times \frac{(1.\mathrm{o}.\mathrm{O})\,\delta' + (\mathrm{I}.\mathrm{o}.\mathrm{O})\,D'}{(1.2.\mathrm{II})\,\delta' + (\mathrm{I}.2.\mathrm{II})\,D'} \times \frac{(2.1.\mathrm{I})\,\delta'' + (\mathrm{II}.1.\mathrm{I})\,D''}{(2.\mathrm{o}.\mathrm{O})\,\delta'' + (\mathrm{II}.\mathrm{o}.\mathrm{O})\,D''} = -1. \tag{7}$$

δ, δ', δ'' are curtate distances drawn from three positions of the earth to an object moving in a great circle about the sun; D, D', D'' are curtate distances drawn from the sun to the earth. We take the origin of coordinates at the centre of the sun, and axes at the earth, wherever it may be, parallel to those at the sun. $(\mathrm{o}.1.2)$ denotes

$$\tan\beta \sin(\alpha'' - \alpha') + \tan\beta' \sin(\alpha - \alpha'') + \tan\beta'' \sin(\alpha' - \alpha),$$

and to get $(\mathrm{O}.1.2)$ we replace α by L, and β by B, and so with the other expressions of (7) in parentheses.

Now in this expression (7) δ cannot be zero, nor δ', δ'', D, D', D''; therefore, if (7) be indeterminate it must have a zero factor common to every one of its terms. It may be noticed that the condition given in the English edition of the *Theoria Motus* that (7) should be indeterminate, although a correct condition as there given, is not the general condition, but a special case to which the general condition reduces when $B = B' = B'' = 0$. Clearing the expresssion (7) of fractions, we have:—

$$\begin{aligned}
&\delta\delta'\delta''\,[(\mathrm{o}.2.\mathrm{II})(1.\mathrm{o}.\mathrm{O})(2.1.\mathrm{I}) + (\mathrm{o}.1.\mathrm{I})(1.2.\mathrm{II})(2.\mathrm{o}.\mathrm{O})]\\
&+ D'\delta\delta''\,[(\mathrm{o}.2.\mathrm{II})(\mathrm{I}.\mathrm{o}.\mathrm{O})(2.1.\mathrm{I}) + (\mathrm{o}.1.\mathrm{I})(\mathrm{I}.2.\mathrm{II})(2.\mathrm{o}.\mathrm{O})]\\
&+ D\delta'\delta''\,[(\mathrm{O}.2.\mathrm{II})(1.\mathrm{o}.\mathrm{O})(2.1.\mathrm{I}) + (\mathrm{O}.1.\mathrm{I})(1.2.\mathrm{II})(2.\mathrm{o}.\mathrm{O})]\\
&+ DD'\delta''\,[(\mathrm{O}.2.\mathrm{II})(\mathrm{I}.\mathrm{o}.\mathrm{O})(2.1.\mathrm{I}) + (\mathrm{O}.1.\mathrm{I})(\mathrm{I}.2.\mathrm{II})(2.\mathrm{o}.\mathrm{O})] \quad (A)\\
&+ D''\delta\delta'\,[(\mathrm{o}.2.\mathrm{II})(1.\mathrm{o}.\mathrm{O})(\mathrm{II}.1.\mathrm{I}) + (\mathrm{o}.1.\mathrm{I})(\mathrm{I}.2.\mathrm{II})(\mathrm{II}.\mathrm{o}.\mathrm{O})]\\
&+ D'D''\delta\,[(\mathrm{o}.2.\mathrm{II})(\mathrm{I}.\mathrm{o}.\mathrm{O})(\mathrm{II}.1.\mathrm{I}) + (\mathrm{o}.1.\mathrm{I})(\mathrm{I}.2.\mathrm{II})(\mathrm{II}.\mathrm{o}.\mathrm{O})]\\
&+ DD''\delta'\,[(\mathrm{O}.2\,\mathrm{II})(1.\mathrm{o}.\mathrm{O})(\mathrm{II}.1.\mathrm{I}) + (\mathrm{O}.1.\mathrm{I})(1.2.\mathrm{II})(\mathrm{II}.\mathrm{o}.\mathrm{O})]\\
&+ DD'D''\,[(\mathrm{O}.2.\mathrm{II})(\mathrm{I}.\mathrm{o}.\mathrm{O})(\mathrm{II}.1.\mathrm{I}) + (\mathrm{O}.1.\mathrm{I})(\mathrm{I}.2.\mathrm{II})(\mathrm{II}.\mathrm{o}.\mathrm{O})]\\
&= 0.
\end{aligned}$$

By actually multiplying out the co-efficient of $\delta\delta'\delta''$ we get for it

$$(0\,.\,1\,.\,2)\,.\left\{\begin{array}{lllll} & \tan {}_{,}\beta' & \tan {}_{,}\beta'' & \sin(L''-L') & \sin(L-a) \\ + & \tan B' & \tan B'' & \sin(a''-a') & \sin(L-a) \\ + & \tan {}_{,}\beta'' & \tan {}_{,}\beta & \sin(L-L'') & \sin(L'-a') \\ + & \tan B'' & \tan B & \sin(a-a'') & \sin(L'-a') \\ + & \tan {}_{,}\beta & \tan {}_{,}\beta' & \sin(L'-L) & \sin(L''-a'') \\ + & \tan B & \tan B' & \sin(a'-a) & \sin(L''-a'') \\ - & \tan {}_{,}\beta' & \tan B'' & \sin(a''-L') & \sin(L-a) \\ - & \tan B' & \tan {}_{,}\beta'' & \sin(L''-a') & \sin(L-a) \\ - & \tan {}_{,}\beta'' & \tan B & \sin(a-L'') & \sin(L'-a') \\ - & \tan B'' & \tan {}_{,}\beta & \sin(L-a'') & \sin(L'-a') \\ - & \tan {}_{,}\beta & \tan B' & \sin(a'-L) & \sin(L''-a'') \\ - & \tan B & \tan {}_{,}\beta' & \sin(L'-a) & \sin(L''-a'') \end{array}\right\}. \qquad (B)$$

By inspection of (A) we see that if we take the co-efficient of $\delta\delta'\delta''$, change 1 into I, and I into 1 throughout, we will get the negative of the co-efficient of $D'\delta\delta''$. But in (B), which is identical with the co-efficient of $\delta\delta'\delta''$, if we change 1 into I, and I into 1, (0 . 1 . 2) becomes (0 . I . 2) and the sign of the second parenthesis is changed, so that, calling U the twelve term expression in the second parenthesis, the negative of the co-efficient of $D'\delta\delta''$ becomes

$$-(0\,.\,\mathrm{I}\,.\,2)\,.\,U,$$

or the co-efficient of $D'\delta\delta''$ becomes

$$(0\,.\,\mathrm{I}\,.\,2)\,.\,U, \qquad (C)$$

so that the co-efficients of $\delta\delta'\delta''$ and $D'\delta\delta''$, each co-efficient being composed of two factors, have the common factor U. The other factor of the co-efficient of $D'\delta\delta''$ is derived from (0 . 1 . 2) by substituting I for 1, corresponding to the introduction of D' into $\delta\delta'\delta''$ in the place of δ'. In the same way, the co-efficient of $D\delta'\delta''$ is

$$(\mathrm{O}\,.\,1\,.\,2)\,.\,U.$$

Thus we get for (A)

$$[(0\,.\,1\,.\,2)+(0\,.\,\mathrm{I}\,.\,2)+(\mathrm{O}\,.\,1\,.\,2)+(\mathrm{O}\,.\,\mathrm{I}\,.\,2)+(0\,.\,1\,.\,\mathrm{II})+(0\,.\,\mathrm{I}\,.\,\mathrm{II})$$
$$+(\mathrm{O}\,.\,1\,.\,\mathrm{II})+(\mathrm{O}\,.\,\mathrm{I}\,.\,\mathrm{II})]\,.\,U=0, \qquad (D)$$

which is the general form of equation (7) when it is cleared of fractions. Now, the expression in the brackets cannot generally be zero, and hence the condition that (D) or (7) should be identically true is

$$U=0,$$

and in this case $\delta, \delta', \delta''$ may have indeterminate values. In the special case

when $B = B' = B'' = 0$, the case which really occurs in practice, U is much simplified, and it is in this simpler form that the condition is given in the English translation.

In article 112 we have derived the equations

$$\begin{aligned}
&(1) \qquad 0 = n(\delta \cos \alpha + D \cos L) - n'(\delta' \cos \alpha' + D' \cos L') \\
&\qquad\qquad + n''(\delta'' \cos \alpha'' + D'' \cos L''), \\
&(2) \qquad 0 = n(\delta \sin \alpha + D \sin L) - n'(\delta' \sin \alpha' + D' \sin L') \\
&\qquad\qquad + n''(\delta'' \sin \alpha'' + D'' \sin L''), \\
&(3) \qquad 0 = n(\delta \tan \beta + D \tan B) - n'(\delta' \tan \beta' + D' \tan B') \\
&\qquad\qquad + n''(\delta'' \tan \beta'' + D'' \tan B'').
\end{aligned}$$

If we regard n, n', n'' as quantities to be determined, in order that they may have values which are not zero, we must have the determinant condition

$$(8) \qquad \begin{vmatrix} \delta \cos \alpha + D \cos L, & \delta'' \cos \alpha'' + D'' \cos L'', & \delta' \cos \alpha' + D' \cos L' \\ \delta \sin \alpha + D \sin L, & \delta'' \sin \alpha'' + D'' \sin L'', & \delta' \sin \alpha' + D' \sin L' \\ \delta \tan \beta + D \tan B, & \delta'' \tan \beta'' + D'' \tan B'', & \delta' \tan \beta' + D' \tan \alpha' \end{vmatrix} = 0,$$

which is equation (8) of article 114. Multiplying out the determinant we have

$$\begin{aligned}
&(0.1.2)\,\delta\delta'\delta'' + (\mathrm{O}.1.2)\,D\delta'\delta'' + (0.\mathrm{I}.2)\,D'\delta\delta'' \\
(8) \qquad &+ (0.1.\mathrm{II})\,D''\delta\delta' + (0.\mathrm{I}.\mathrm{II})\,D'D''\delta + (\mathrm{O}.1.\mathrm{II})\,DD''\delta' \\
&+ (\mathrm{O}.\mathrm{I}.2)\,DD'\delta'' + (\mathrm{O}.\mathrm{I}.\mathrm{II})\,DD'D'' = 0.
\end{aligned}$$

We see that equation (8) does not become indeterminate with (7) and may be used when (7) fails.

In article 114 we have given the following equations:—

$$\begin{aligned}
&(9) \quad 0 = n[(0.1.2)\,\delta + (\mathrm{O}.1.2)\,D] - n'(\mathrm{I}.1.2)\,D' + n''(\mathrm{II}.1.2)\,D'', \\
&(10) \quad 0 = n(0.\mathrm{O}.2)\,D - n'[(0.1.2)\,\delta' + (0.\mathrm{I}.2)\,D'] + n''(0.\mathrm{II}.2)\,D'', \\
&(11) \quad 0 = n(0.1.\mathrm{O})\,D - n'(0.1.\mathrm{I})\,D' + n''[(0.1.2)\,\delta'' + (0.1.\mathrm{II})\,D''.]
\end{aligned}$$

Suppose $(0.1.2)$ equals zero, then we cannot get from these equations values of $\delta, \delta', \delta''$. Now, if we draw from the sun radii vectores parallel to the radii vectores drawn from the three positions of the earth to the planet, the condition

$$(0.1.2) = 0$$

means that the three radii vectores so drawn from the sun lie in a plane. Suppose we take this plane for the plane of x, y in the system of co-ordinate axes at the sun. Then $\beta = \beta' = \beta'' = 0$, and we have for equation (9), remembering that $(0.1.2) = 0$,

$$nZ - n'Z' + n''Z'' = 0.$$

Also (10) and (11) reduce to this same equation. And since $Z = z, Z' = z', Z'' = z''$,

from the way in which we have chosen our axes, (9), (10), and (11) each reduce to

$$nz - n'z' + n''z'' = 0, \qquad (E)$$

which is a known, necessary relation (see article 112), and does not depend upon the position of the earth at all. And merely by transformation of co-ordinates (9), (10), and (11) must be derived from (E).

We have also derived in article 114 the following equations:—

$$(4)\quad 0 = n\,[(0.2.\mathrm{II})\delta + (\mathrm{O}.2.\mathrm{II})D] - n'\,[(1.2.\mathrm{II})\delta' + (\mathrm{I}.2.\mathrm{II})D']$$
$$(5)\quad 0 = n\,[(0.1.\mathrm{I})\ \delta + (\mathrm{O}.1.\mathrm{I})\ D] + n''[(2.1.\mathrm{I})\ \delta'' + (\mathrm{II}.1.\mathrm{I})D'']$$
$$(6)\quad 0 = n'[(1.0.\mathrm{O})\delta' + (\mathrm{I}.0.\mathrm{O})D'] - n''[(2.0.\mathrm{O})\delta'' + (\mathrm{II}.0.\mathrm{O})D'']$$

Likewise here, we come back merely to necessary relations if in these equations a co-efficient of δ, or δ', or of δ'' becomes equal to zero. Thus, suppose in (4) the co-efficient of δ, $(0.2.\mathrm{II}) = 0$. Then we have lying in a plane the radius vector drawn from the sun to the third position of the earth, the line drawn from the sun parallel to the radius vector drawn from the first position of the earth to the planet, and the line drawn from the sun parallel to the radius vector drawn from the third position of the earth to the planet. Take the plane containing these three lines as the plane of x, y in the system of co-ordinate axes at the sun. Then we have $\beta = \beta'' = B'' = 0$, and equation (4) reduces to

$$nZ - n'z' - n'Z' = 0;$$

also, $z = Z$, $z'' = Z'' = 0$, hence this equation reduces to

$$nz - n'z' - n'Z' = 0 \qquad (F)$$

Now, we have as a necessary relation

$$nz - n'z' + n''z'' = 0$$

or since $z'' = 0$

$$nz - n'z' = 0.$$

Hence we have from (F) $Z' = 0$, or the second position of the earth as well as the third lies in the plane of x, y, therefore the plane of x, y passing through the sun and two positions of the earth is the orbit of the earth. Hence $z = Z = 0$, also since $z'' = 0$, we have from the necessary relation

$$nz - n'z' + n''z'' = 0$$

$z' = 0$, or the plane of the orbit of the planet coincides with the ecliptic. We get the same result if we take any other co-efficient of δ, or δ', or of δ'' equal to zero.

The formulæ of this article are important, since they give the relations between the distances of a planet from the earth at the times of three observations, and it is on the determination of these distances that a knowledge of the

orbit of the planet depends. Equation (8) shows that if two of the distances are known the third can be computed. Equation (5) furnishes the relation between the curtate distances used by Olbers in his well known method of computing the orbit of a comet.

SOLUTIONS OF EXERCISES.

10

REQUIRED the length of a thread wrapped spirally round the frustum of a given cone, the distance between the spires along the slant height being constant.

[*A. B. Nelson.*]

SOLUTION.

The development of the thread on a plane obtained by rolling the cone on the plane will be a spiral of Archimedes, the length of which is well known.

[*DeVolson Wood.*]

12

THE RESULT

$$-\frac{p^2q^2 + 4p^3r - 8q^3 + 2pqr + 9r^3}{(r - pq)^2}$$

is given as the equivalent of the function

$$\left(\frac{\beta - \gamma}{\beta + \gamma}\right)^2 + \left(\frac{\gamma - \alpha}{\gamma + \alpha}\right)^2 + \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2,$$

where α, β, γ are the roots of the cubic

$$x^3 + px^2 + qx + r = 0.$$

Is this result correct? [*A. Hall.*]

SOLUTION.

The result is *not* correct. The symmetric function $\Sigma\left(\frac{\beta - \gamma}{\beta + \gamma}\right)^2$ of the roots α, β, γ of the cubic $x^3 + px^2 + qx + r = 0$, expressed in terms of the co-efficients, is

$$= -\frac{-3p^2q^2 + 4p^3r + 4q^3 + 2pqr + 9r^2}{(r - pq)^2}.$$

Indeed, we have

$$\Sigma \left(\frac{\beta-\gamma}{\beta+\gamma}\right)^2 = \frac{\Sigma(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2}{(\beta+\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2},$$

say $= \frac{N}{\Delta}$, and on the other hand the well known relations $\Sigma\alpha = -p$, $\Sigma\beta\gamma = q$, $\alpha\beta\gamma = -r$, where Σ always indicates summation with respect to the three roots.

For the denominator Δ we find:

$$\begin{aligned}(\beta+\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2 &= (\Sigma\alpha-\alpha)^2(\Sigma\alpha-\beta)^2(\Sigma\alpha-\gamma)^2 \\ &= (p+\alpha)^2(p+\beta)^2(p+\gamma)^2 \\ &= (p^3+\Sigma\alpha\,.\,p^2+\Sigma\alpha\beta\,.\,p+\alpha\beta\gamma)^2 \\ &= (r-pq)^2.\end{aligned}$$

To express the numerator N in terms of p, q, r, we observe first that the "weight" of the function is 6. The general form of the expression will therefore be

$$Ap^2q^2 + Bp^3r + Cq^3 + Dpqr + Er^2,$$

since this contains all possible combinations of p, q, r of the weight 6. This agrees with the general form of the expression as stated in the problem, with the exception of the last term, where r^3 is probably a misprint for r^2.

The most expedient way to determine the numerical co-efficients seems to be the following:—

Assume any five cubic equations whose roots are known, and calculate for every one the values of $\Sigma(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2$ and of the co-efficients of A, B, C, D, E in the general form. This gives five equations sufficient for the determination of A, B, C, D, E.

Taking for instance the equations

$$\begin{aligned}
&0=(x-1)^2(x+2) &&= x^3-3x+2, &&(1)\\
&0=(x-1)(x-2)(x+3) &&= x^3-7x+6, &&(2)\\
&0=(x-1)^2(x+1) &&= x^3-x^2-x+1, &&(3)\\
&0=(x-1)^3 &&= x^3-3x^2+3x-1, &&(4)\\
&0=(x-1)(x-2)(x-3) &&= x^3-6x^2+11x-6; &&(5)
\end{aligned}$$

we have

$\alpha =$	$\beta =$	$\gamma =$	$p =$	$q =$	$r =$	
$+1$	$+1$	-2	0	-3	$+2$	(1)
$+1$	$+2$	-3	0	-7	$+6$	(2)
$+1$	$+1$	-1	-1	-1	$+1$	(3)
$+1$	$+1$	$+1$	-3	$+3$	-1	(4)
$+1$	$+2$	$+3$	-6	$+11$	-6	(5)

Substituting in the equation

$$Ap^2q^2 + Bp^3r + Cq^3 + Dpqr + Er^2 = \Sigma(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2,$$

we have

$$\begin{array}{rrrrrl} & & -\ 27C & & +\ 4E = & +\ 72, \\ & & -\ 343C & & +\ 36E = & +\ 1048, \\ +\ A - & B - & C + & D + & E = & 0, \\ +\ 81A + & 27B + & 27C + & 9D + & E = & 0, \\ +\ 4356A + & 1296B + & 1331C + & 396D + & 36E = & +\ 1444; \end{array}$$

from which we find

$$A = +3,\ B = -4,\ C = -4,\ D = -2,\ E = -9.$$

This gives

$$\Sigma(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2 = +3p^2q^2 - 4p^3r - 4q^3 - 2pqr - 9r^2.$$

The direct derivation of this expression from the fundamental relations $\Sigma\alpha = -p$, $\Sigma\beta\gamma = q$, $\alpha\beta\gamma = -r$ is somewhat lengthy. It may be effected in the following way:—

$$\begin{aligned}\Sigma(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2 &= \Sigma(\beta-\gamma)^2(\alpha^2+\beta\gamma+\gamma\alpha+\alpha\beta)^2 \\ &= \Sigma(\beta-\gamma)^2(\alpha^2+q)^2 \\ &= \Sigma\alpha^4(\beta-\gamma)^2 + 2q\,\Sigma\alpha^2(\beta-\gamma)^2 + q^2\,\Sigma(\beta-\gamma)^2\end{aligned}$$

To calculate these three terms we want the following relations:—

$$(\Sigma\alpha)^2 = p^2 = \Sigma\alpha^2 + 2\Sigma\beta\gamma,$$
$$\therefore\ \Sigma\alpha^2 = p^2 - 2q;$$

$$(\Sigma\beta\gamma)^2 = q^2 = \Sigma\beta^2\gamma^2 + 2\Sigma\alpha^2\beta\gamma,$$
$$\therefore\ \Sigma\beta^2\gamma^2 = q^2 - 2pr;$$

$$\begin{aligned}\Sigma\alpha^2.\Sigma\beta^2\gamma^2 &= (p^2-2q)(q^2-2pr) \\ &= \Sigma\alpha^4(\beta^2+\gamma^2) + 3\alpha^2\beta^2\gamma^2,\end{aligned}$$
$$\therefore\ \Sigma\alpha^4(\beta^2+\gamma^2) = p^2q^2 - 2q^3 - 2p^3r + 4pqr - 3r^2;$$

$$\Sigma\alpha\,.\,\Sigma\beta\gamma = -pq = \Sigma\beta^2\gamma + 3\alpha\beta\gamma,$$
$$\therefore\ \Sigma\beta^2\gamma = 3r - pq;$$

$$\Sigma\alpha\,.\,\Sigma\alpha^2 = -p(p^2-2q) = \Sigma\alpha^3 + \Sigma\beta^2\gamma,$$
$$\therefore\ \Sigma\alpha^3 = -p^3 + 3pq - 3r;$$

$$\begin{aligned}\alpha\beta\gamma\,.\,\Sigma\alpha^3 &= -r(3pq - p^3 - 3r) = \Sigma\alpha^4\beta\gamma \\ &= -3pqr + p^3r + 3r^2.\end{aligned}$$

With the aid of these expressions we find:—

$$\Sigma(\beta-\gamma)^2 = 2\Sigma\alpha^2 - 2\Sigma\beta\gamma = 2(p^2-3q),$$

$$\Sigma\alpha^2(\beta-\gamma)^2 = 2\,\Sigma\beta^2\gamma^2 - 2\,\Sigma\alpha^2\beta\gamma = 2\,(q^2-3pr),$$
$$\Sigma\alpha^4(\beta-\gamma)^2 = \Sigma\alpha^4(\beta^2+\gamma^2) - 2\Sigma\alpha^4\beta\gamma$$
$$= p^2q^2 - 2q^3 - 4p^3r + 10pqr - 9r^2;$$
$$\therefore\ \Sigma(\beta-\gamma)^2(\gamma+\alpha)^2(\alpha+\beta)^2 = +\,3p^2q^2 - 4p^3r - 4q^3 - 2pqr - 9r^2,$$

as above.

[*Alexander Ziwet.*]

14

A FAISCEAU of parabolas can be drawn having the pole of a cardioid as the common focus, all passing through one point and all cutting the cardioid at right angles. [*H. A. Newton.*]

SOLUTION.

The equation of a cardioid is

$$r = m\,(1-\cos\theta);$$

and that of a parabola whose focus is at the pole of the cardioid is

$$r = \frac{p}{1-\cos(\theta-a)}.$$

When these equations are differentiated the former yields

$$r\,\frac{d\theta}{dr} = \frac{1-\cos\theta}{\sin\theta},$$

and the latter

$$r\,\frac{d\theta}{dr} = -\,\frac{1-\cos(\theta-a)}{\sin(\theta-a)},$$

as expressions for the tangents of the angles made by the respective curves with the radius vector. Hence, if θ' represent the vectorial angle belonging to that radius vector which meets the two curves at angles differing by 90°, we have

$$\frac{1-\cos\theta'}{\sin\theta'}\cdot\frac{1-\cos(\theta'-a)}{\sin(\theta'-a)} = 1;$$

and if the same radius vector meet the curves at their intersection, we have also

$$m\,(1-\cos\theta') = \frac{p}{1-\cos(\theta'-a)}.$$

The former of these conditions is satisfied if

$$\theta' + (\theta'-a) = 180°,$$

since in this case it reduces to

$$1-\cos^2\theta' = \sin^2\theta'.$$

The value of θ', as thus determined, is

$$\theta' = 90^\circ + \tfrac{1}{2}a,$$

and this value, substituted in the second equation of condition, yields

$$\begin{aligned} p &= m\,(1 + \sin\tfrac{1}{2}a)\,(1 - \sin\tfrac{1}{2}a) \\ &= m\cos^2\tfrac{1}{2}a. \end{aligned}$$

Hence the equation

$$r = \frac{m\cos^2\frac{1}{2}a}{1 - \cos(\theta - a)}$$

represents a parabola meeting the cardioid at right angles: and any assigned value of a determines one parabola. But in any such parabola, when $\theta = 180^\circ$, $r = \frac{1}{2}m$, independently of the value of a, since $\dfrac{\cos^2\frac{1}{2}a}{1 + \cos a} = \dfrac{1}{2}$; hence all of these parabolas pass through a single point on the axis of the cardioid, as was to be shown. [*F. H. Loud.*]

17

IN HIS WORK, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, p. 65, Grassmann says: "Lagrange führt in seiner *Méc. Anal.*, p. 14 der neuen Ausgabe, einen Satz von Varignon an, dessen er sich zur Verknüpfung der verschiedenen Principien der Statik bedient. * * * * Dieser Satz ist, wie sich sogleich zeigen wird, unrichtig."

In what way is this theorem incorrect as used by Todhunter and others?

[*Asaph Hall.*]

SOLUTION.

Varignon's theorem as stated by Lagrange at the place above referred to is substantially as follows: If from a point in the plane of a parallelogram perpendiculars be let fall on two contiguous sides and on the co-initial diagonal, then the product of the diagonal into its perpendicular is equal to the sum or to the difference of the products of the sides into their respective perpendiculars *according as the point lies outside or inside the parallelogram.* The final clause is not correct, and, as Grassmann has pointed out, should be: *according as the point lies outside or inside the two angular spaces containing the diagonal and its prolongation.*

Varignon himself, of course, gives his theorem correctly.* But the manner in which he treats it is very lengthy and cumbersome: the enunciation of the theorem alone occupies a whole folio page and the demonstration is broken up into as many special cases as there are different positions for the point, and illustrated by over a dozen different diagrams.

*See his *Nouvelle Mécanique ou Statique*, etc. Paris, 1725. Vol. I. p. 84 foll.

Lagrange, too, must have had in his mind the correct form of the proposition and his words "hors du parallélogramme" are only a slip of the pen that may be easily pardoned. This will appear from the developments with which he follows it, where he remarks that the theorem is true if the sides and diagonal of the parallelogram are moved within the lines in which they lie.

Todhunter, in his *Analytical Statics* (4th ed., Lon. 1874), p. 53, states the theorem in the following simple and correct form: "The algebraical sum of the moments of two component forces with respect to any point in the plane containing the two forces is equal to the moment of the resultant of the two forces." Similarly in his *Mechanics for Beginners* (5th ed., Lon. 1880), p. 40–1. In Geo. M. Minchin's *Statics* (2nd ed., Oxf. 1880), p. 67–8, we find: "The virtual work of a force is equal to the sum of the virtual works of its components, rectangular or oblique."

In these forms of the theorem, as statically expressed, any ambiguity disappears and it becomes impossible to repeat Lagrange's incorrectness, simply because the algebraical sign of "moments" and "works" is fixed by definition and leaves no room for uncertainty.

This precision and simplicity of statement however may be reached just as well in the purely *geometrical* form of the proposition, which ought to be made the basis of its applications to mechanics. It is only necessary to adopt the notion of plane areas affected by sign, or, more correctly speaking, to define the area of a triangle as a vector quantity. It will then be readily seen that Varignon's theorem is equivalent to the fundamental proposition in the theory of trilinear co-ordinates $ap + bq + cr = 2\Delta$.* Hamilton, Grassmann, Moebius, and others have clearly shown the advantages to be derived from treating areas as vectors and "geometrical" products, and perhaps no other proposition of such an elementary character could so forcibly illustrate the superiority of modern geometrical methods. In this respect it will prove interesting to contrast Varignon's presentation, which must be conceded to be a fine example of strictly Euclidean geometry, with W. K. Clifford's simple and elegant treatment of the same theorem, by modern methods, in his admirable little treatise on Dynamics (Part I: *Kinematic*, Lon. 1878), p. 92 foll.

[*Alexander Ziwet.*]

*See *Salmon's Higher Plane Curves*, Chap. I.

EXERCISES.

19

SHOW geometrically (without using the calculus) that the asymptote of the hyperbolic spiral, $r = a : \theta$, is parallel to the initial line and distant a above it.

[*L. G. Barbour.*]

20

A TUNNEL section, consisting of a rectangle surmounted by a semicircle, is required to accommodate a rectangle of breadth B and height H. Show how to determine its proportions so that its (1) area, (2) perimeter shall be least.

[*William M. Thornton.*]

21

IF THE central force on a body moving in a parabola, latus rectum $4m$, were to cease acting at the vertex and continue interrupted till the body had described an angle of 60° about the focus; determine the orbit it would afterwards describe.

[*William Hoover.*]

22

THE dome of the rotunda of the University of Virginia is spherical. The length of its meridian section is 85'.2 and the girth of its base is 214'. From these data it is required to compute the radius and the surface of the dome.

[*William M. Thornton.*]

23

IF TWO parabolas P and P' cut each other at right angles at a point A on a third parabola P'', all three parabolas having a common focus F, and if the tangent line drawn at A to P'' cut P and P' in B and C, then will one parabola that passes through B and C and has F for its focus cut P'' at right angles.

[*H. A. Newton.*]

24

The curves of the family

$$\left(\frac{C}{r}\right) = \cos(p + n\theta),$$

where n is a parameter, all pass through a fixed point and cut orthogonally the fixed curve

$$\left(\frac{r}{c}\right)^n = \cos n\theta,$$

provided $C^n = \frac{1}{2}c^n \cos p$.

[Generalization of 14.] [*Alfred C. Lane.*]

25

An elastic ring of radius a is placed gently on a smooth paraboloid of revolution, whose axis is vertical; find, by use of the principle of energy, the lowest position to which the ring will descend, and its position of static equilibrium. [*R. D. Bohannon.*]

NOTES.

3

Mr. Joseph B. Mott, of Worthington, Minn., sends, in connexion with an invalid deduction of the logarithmic series, several ingenious combinations for the computation of logarithms of primes. We note the following: —

$$\log 11 = 1 + \frac{1}{2}\log 2 - 2\log 3 + \log 7$$
$$+ M\left\{\frac{1}{19601} + \frac{1}{3}\cdot\frac{1}{19601^3} + \frac{1}{5}\cdot\frac{1}{19601^5} + \dots\right\}.$$
$$\log 13 = \frac{3}{2}\log 2 + \frac{3}{2}\log 11 - \log 3 - \frac{1}{2}\log 7$$
$$- M\left\{\frac{1}{21295} + \frac{1}{3}\cdot\frac{1}{21295^3} + \frac{1}{5}\cdot\frac{1}{21295^5} + \dots\right\}.$$

Like results are given for log 17 and log 19; and the logarithms of these and other primes are computed with considerable facility to thirty-two places.

4

Perhaps no modern geometer has fallen upon an easier and more rapid process for such computations than that indicated by Newton (*Epistola posterior ad Oldenburgium*, Oct. 24, 1676; Opuscula 1, 328). Newton computes for $x = 0,1$; $0,2$; $0,01$; $0,02$; $0,001$; $0,002$ the values of

$$\log\sqrt{\frac{1+x}{1-x}} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots,$$
$$\log\sqrt{\frac{1}{1-x^2}} = x^2 + \frac{1}{4}x^4 + \frac{1}{6}x^6 + \dots;$$

whence by the aid of the simple interpolation formula

$$\log n = \log(n-x) + d\left(1 + \frac{x}{2n} + \frac{x^3}{12n^3} + \dots\right),$$

where

$$d = \log(n + x) - \log(n - x),$$

are rapidly calculated the logarithms of the integers 80–90, 980–990, 9980–9990. These give the logarithms of all primes up to 100.

As editorial comment on such diversions we may be permitted to quote Newton's confession: "Pudet dicere," he says, "ad quot figurarum loca has computationes otiosus eo tempore produxi. Nam tunc sane nimis delectabar inventis hisce."

5

THE equation of a conic may always be written in the form $a\gamma = k\beta^2$, where $a = 0$, $\gamma = 0$ represent the equations of any two tangents of the curve, and $\beta = 0$ the equation of their chord of contact. If for a, γ we take the asymptotes, β will be the right line at infinity; i. e. reduces to a constant. Passing to Cartesian co-ordinates and taking as axis of x one of the bisectors of the angles between the asymptotes, we shall have $a = y - ax - b$, $\gamma = y + ax + b$, and hence the equation of the conic

$$y^2 - (ax + b)^2 = c, \text{ or } y^2 = a^2x^2 + 2abx + b'^2,$$

which is Prof. Nicholson's form.* This derivation shows that, geometrically, Prof. Nicholson's criterion for the discrimination of the three species of conics is simply the reality or non-reality of the asymptotes, as was to be expected.

[*Alexander Ziwet.*]

6

THERE are in mathematics not a few formulæ which bear the names of others than those who first actually or virtually discovered them. Well known examples are Maclaurin's theorem, Cardan's rule, Demoivre's theorem. In 1879 Prof. E. Schering of Göttingen added Lagrange's interpolation-formula to this list. This formula is due to Waring, who gave it in a paper on "Problems concerning Interpolations," read before the Royal Society of London, Jan. 9th, 1779. In the works of Lagrange, edited by Serret, there are given (Vol. VII. p. 285) the lectures on elementary mathematics which Lagrange delivered in Paris in 1795, and in which the interpolation-formula occurs. Serret remarks (Vol. VII. p. 183) that these Lectures were prepared during 1794–95, and that seventeen years later (1812), upon the recommendation of Lagrange, they were reprinted in the *Journal de l'École Polytechnique*, Vol. II. p. 417.

I do not know that any English or American mathematical journal has called attention to this matter. It is but right to give Waring all honor that is due him.

[*R. D. Bohannan.*]

*See *Annals of Mathematics* for May, 1884.

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